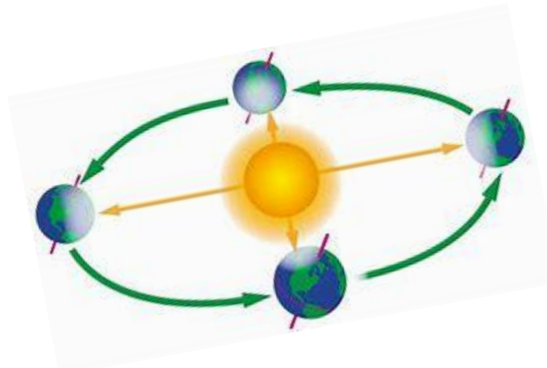
CH 34 – THE ELLIPSE

❑ INTRODUCTION



An ***ellipse*** is oval-shaped, and may be described as a circle that never quite made it. It's the shape of orbits. For example, the orbit of the Earth around the Sun is an ellipse, and the orbits of many satellites (including our Moon) revolving around the Earth are elliptical. The elliptical shape is also used in certain gears, and is the key behind a medical procedure called *lithotripsy* (Greek for stone-crushing) – the use of shock waves to break up kidney stones.

❑ FINDING INTERCEPTS

In this section we recall the procedures for finding intercepts; the practice will help us in the ellipse problems which follow.

EXAMPLE 1: **Find all the intercepts of the graph**

$$\mathbf{49x^2 + 9y^2 = 441}$$

Solution: To find any x-intercepts, we set $y = 0$ and solve for x:

$$49x^2 + 9y^2 = 441 \Rightarrow 49x^2 + 9(\mathbf{0})^2 = 441$$

$$\Rightarrow 49x^2 = 441 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

Thus, the x-intercepts are **(3, 0)** and **(–3, 0)**.

To find any y-intercepts, we set $x = 0$ and solve for y:

$$49x^2 + 9y^2 = 441 \;\Rightarrow\; 49(\mathbf{0})^2 + 9y^2 = 441$$

$$\Rightarrow\; 9y^2 = 441 \;\Rightarrow\; y^2 = 49 \;\Rightarrow\; y = \pm 7$$

Therefore, the y-intercepts are **(0, 7)** and **(0, –7)**.

Homework

1. Find all the **intercepts** of each graph:

a. $3x + 5y = 30$ b. $y = 4x + 20$ c. $x = 3$

d. $y = -4$ e. $y = x^2$ f. $y = x^2 - x - 6$

g. $x^2 + y^2 = 121$ h. $x^2 + y^2 = 50$ i. $9x^2 + y^2 = 9$

j. $x^2 + 36y^2 = 36$ k. $81x^2 + 25y^2 = 2025$ l. $x^2 + 4y^2 = 16$

m. $x^2 - y^2 = 100$ n. $y^2 - x^2 = 121$ o. $y = x^2 + x + 1$

❑ *ELLIPSES (WITH CENTER AT THE ORIGIN)*

<u>EXAMPLE 2:</u> **Graph the ellipse:** $\dfrac{\mathbf{x^2}}{\mathbf{4}} + \dfrac{\mathbf{y^2}}{\mathbf{25}} = \mathbf{1}$

<u>Solution:</u> Let's begin by finding all the intercepts of the ellipse. These points and a few others will convince us that the graph is indeed elliptical.

***x*-intercepts:** Set $y = 0$ to get

$$\frac{x^2}{4} + \frac{\mathbf{0}^2}{25} = 1 \;\Rightarrow\; \frac{x^2}{4} = 1 \;\Rightarrow\; x^2 = 4 \;\Rightarrow\; x = \pm\sqrt{4} = \pm 2$$

and so the x-intercepts are **(2, 0)** and **(–2, 0)**.

y-intercepts: Set $x = 0$ to get

$$\frac{\mathbf{0}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = 1 \;\Rightarrow\; y^2 = 25 \;\Rightarrow\; y = \pm\sqrt{25} = \pm 5$$

and thus the y-intercepts are **(0, 5)** and **(0, –5)**.

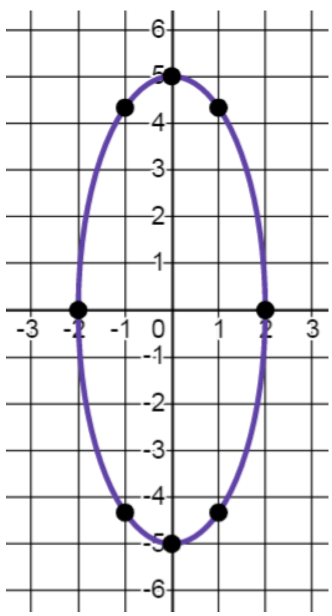
Additional points: Let $x = 1$. Then

$$\frac{\mathbf{1}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{1}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = \frac{3}{4} \;\Rightarrow\; y^2 = \frac{75}{4}$$

$$\Rightarrow\; y = \pm\sqrt{\frac{75}{4}} \;\Rightarrow\; y = \pm\sqrt{18.75} \approx \pm 4.33$$

This yields the points **(1, 4.33)** and **(1, –4.33)**.

Note that if we let $x = -1$, we obtain the same y-values. Thus, two additional points are **(–1, 4.33)** and **(–1, –4.33)**. We now have a total of eight points we can plot. When we do, we get the graph of the ellipse:



Note: The **center** of the ellipse is the origin.

A Final Note:

From the graph we know that an x-value of 3, for example, should not be allowed in the formula (since there's no point on the ellipse whose x-coordinate is 3). But how can we verify this fact before we graph anything? We'll let $x = 3$ in the ellipse equation and see what happens:

$$\frac{x^2}{4}+\frac{y^2}{25} = 1 \overset{x=3}{\Rightarrow} \frac{\mathbf{3}^2}{4}+\frac{y^2}{25} = 1 \Rightarrow \frac{9}{4}+\frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{y^2}{25} = 1-\frac{9}{4} \Rightarrow \frac{y^2}{25} = -\frac{5}{4} \Rightarrow y^2 = -\frac{125}{4}$$

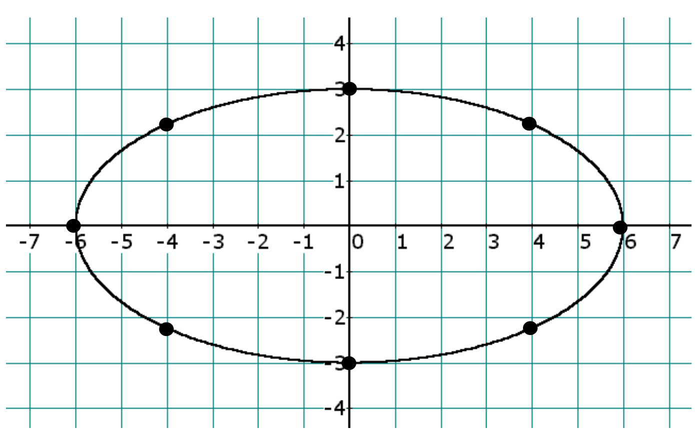
$$\Rightarrow \quad y = \pm\sqrt{-\frac{125}{4}}$$, which are <u>not</u> real numbers.

Since letting $x = 3$ resulted in no real-number solution for y, we conclude that no point on the graph will have an x-coordinate of 3, and thus 3 is not an allowable x-value.

<u>EXAMPLE 3:</u> **Graph the ellipse:** $\mathbf{\frac{x^2}{36}+\frac{y^2}{9} = 1}$

<u>Solution:</u> Since this equation is very similar to the previous one, it's likely that its graph is an ellipse with center at the origin. You can calculate the intercepts to be **(6, 0)**, **(–6, 0)**, **(0, 3)**, and **(0, –3)**.

For extra "resolution," we can find four more points by letting $x = \pm 4$, and then calculating y to be $\pm\sqrt{5}$. We thus get four points with the approximate values (**4, 2.24**), (**4, –2.24**), (**–4, 2.24**), and **(–4, –2.24)**. [Be sure you can use a calculator to determine the decimal approximations.] Here's the graph:



<u>EXAMPLE 4:</u> **Graph the ellipse: $16x^2 + y^2 = 16$**

<u>Solution:</u> Is this an ellipse? It doesn't have the same form as the equations in Examples 2 and 3. Let's find the intercepts, then a few other points, and see what we get.

By letting $x = 0$, we get y-values of ± 4. By letting $y = 0$, we get x-values of ± 1. Therefore, the intercepts are

(1, 0), **(–1, 0)**, **(0, 4)**, and **(0, –4)**

Now we'll let $x = 0.5$. This will give us

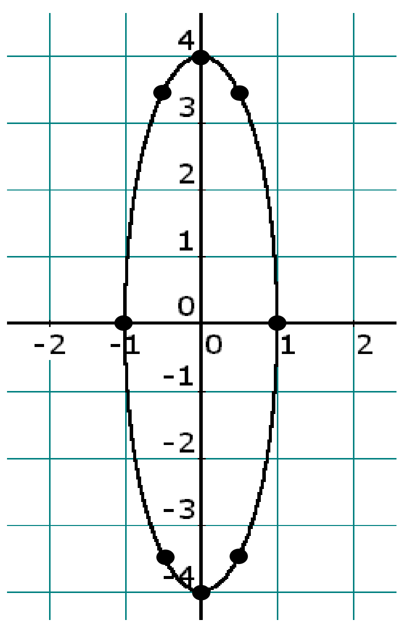
$$16(\mathbf{0.5})^2 + y^2 = 16 \Rightarrow 16(0.25) + y^2 = 16$$

$$\Rightarrow \quad 4 + y^2 = 16 \Rightarrow y^2 = 12 \Rightarrow y = \pm\sqrt{12} \Rightarrow y \approx \pm 3.46$$

Similarly, letting $x = -0.5$ will give the same y-values: ± 3.46. We now have four additional points to graph:

(0.5, 3.46), **(0.5, –3.46)**, **(–0.5, 3.46)**, and **(–0.5, –3.46)**

Let's plot all eight of our points — and sure enough, we get another ellipse:



Homework

2. Prove that no point with an x-coordinate of 12 will lie on the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

3. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

   a. Find all four intercepts.

   b. Find two more points on the ellipse by letting $x = 2$.

   c. Find two more points on the ellipse by letting $x = -3$.

   d. Sketch the ellipse.

Graph each ellipse by plotting the four intercepts and four additional points:

4. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

5. $x^2 + \frac{y^2}{49} = 1$

6. $x^2 + 9y^2 = 9$

7. $\frac{x^2}{81} + \frac{y^2}{9} = 1$

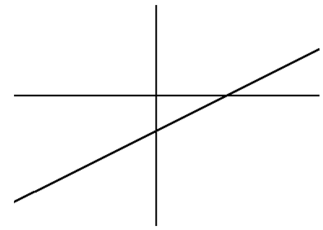
8. $\frac{x^2}{49} + \frac{y^2}{100} = 1$

9. $25x^2 + 4y^2 = 100$

❑ *Lines, Parabolas, Circles, and Ellipses*

Now that we've covered these four shapes in this course, it's the right time to consider the question: Given an equation of one of these four graphs, how can we tell (without graphing) which shape it will be?

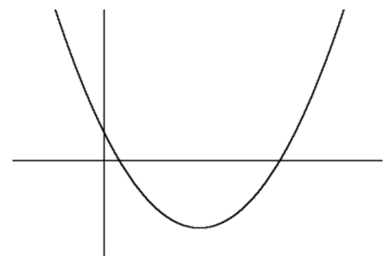
The answer depends on how many of the two variables x and y are squared in the equation. In a ***line***, neither variable is squared; for example,



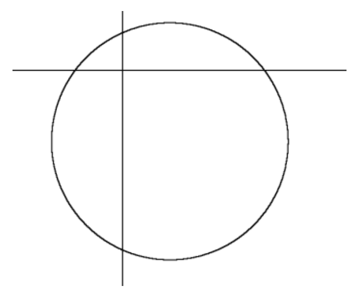
$$3x - 4y = 10 \qquad x = -3 \qquad y = 7$$

are the equations of lines.

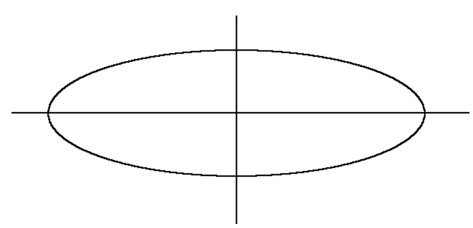
If exactly one variable is squared, then it's a ***parabola***. In this course, a typical parabola might have the equation $y = 2x^2 - 5x + 3$, whose graph opens up. It's also possible that the y is squared, like $x = y^2 + 3$, but this graph is a sideways parabola, but will not be covered until Chapter 45 – Epilogue.



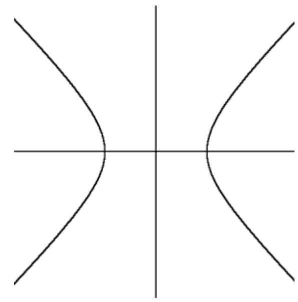
An equation with both variables squared, such as $x^2 + y^2 = 25$, graphed out to be a ***circle*** with center at the origin and radius 5. The equation $(x-2)^2 + (y+7)^2 = 20$ also is a circle, since it's pretty clear that, if expanded, both variables are being squared. In each circle, the coefficients of the two squared terms are the same, namely 1.



As for the ***ellipse***, a typical equation was $9x^2 + 4y^2 = 36$, or perhaps something like $\frac{x^2}{25} + \frac{y^2}{4} = 1$. In these equations, both variables are squared, so why aren't they circles? Here's the answer: In a circle, the coefficients of the squared terms are always the same; in an ellipse, they're different, but have the same sign. In the first ellipse formula above, the coefficients are 9 and 4 (both positive); in the second ellipse, the coefficients are $\frac{1}{25}$ and $\frac{1}{4}$ (again, both positive). Note that the equation $7x^2 + 7y^2 = 30$ is a actually a circle because both variables are squared and the coefficients are the same. In fact, if you divide



each side of the equation by 7, it might be more clear that it's indeed a circle.



We close the chapter with a look at the equation $x^2 - y^2 = 6$. It's not a line because both variables are squared. It's not a parabola for precisely the same reason. It's not a circle since the coefficients (1 and –1) are not the same. And it's not an ellipse because the coefficients don't have the same sign. In other words, it's nothing we've come across in this course. It's called a ***hyperbola*** (hy-PER-bo-la), and will be dealt with in Chapter 45 – Epilogue.

Homework

10. For each formula, determine whether its graph is a line, a parabola, a circle, or an ellipse.

a. $3x - 4y = 22$

b. $x = -9$

c. $y - 7 = 0$

d. $y = -x^2 + 13x$

e. $x^2 + y^2 = 7$

f. $2x^2 + 3y^2 = 23$

g. $(x - 2)^2 + y^2 = 9$

h. $\frac{x^2}{5} + \frac{y^2}{6} = 1$

i. $\frac{x^2}{23} + \frac{y^2}{2} = 1$

j. $17x^2 + 22y^2 = 44$

k. $3x + y^2 - 6 = 0$

l. $(x + 1)^2 + (y - 2)^2 = 29$

m. $y = 17x - 19$

n. $x^2 - y^2 = 12$

Practice Problems

11. Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{100} = 1$.

    a. Prove that $x = 0$ is allowed in the formula.

    b. Prove that $x = 4$ is not allowed in the formula.

12. Find all the intercepts of $25x^2 + 4y^2 = 100$.

13. True/False:

    a. The ellipse $25x^2 + 4y^2 = 100$ has four intercepts.

    b. The graph of $9x^2 - 16y^2 = 144$ is an ellipse.

    c. The graph of $10x^2 = y^2 + 100$ is an ellipse.

14. Matching:

_____ $9x^2 - 18y + 7x + 9y^2 + 1 = 0$	A. line
_____ $18x - \pi y = 171.45$	B. parabola
_____ $y = \sqrt{2}\,x^2 - 200x + \frac{17}{2}$	C. circle
_____ $\frac{x^2}{17} + \frac{y^2}{5} = 1$	D. ellipse
_____ $y = -9$	E. hyperbola

15. Graph the ellipse by plotting the four intercepts and four additional points: $\frac{x^2}{9} + \frac{y^2}{25} = 1$

16. Graph the ellipse by plotting the four intercepts and four additional points: $9x^2 + 16y^2 = 576$

Solutions

1. a. Setting $y = 0$ gives $x = 10$. Setting $x = 0$ gives $y = 6$. The intercepts are (10, 0) and (0, 6).

b. (–5, 0) and (0, 20)

c. (3, 0)

d. (0, –4)

e. (0, 0) is both the x- and y-intercept.

f. Setting $y = 0$ yields the quadratic equation $x^2 - x - 6 = 0$, which factors into $(x - 3)(x + 2) = 0$, whose solutions are $x = 3, -2$. The x-intercepts are therefore (3, 0) and (–2, 0). The y-intercept is (0, –6).

g. There are four intercepts: (±11, 0) and (0, ±11).

h. $(\pm 5\sqrt{2}, 0)$ and $(0, \pm 5\sqrt{2})$

i. (±1, 0) and (0, ±3)

j. (±6, 0) and (0, ±1)

k. (±5, 0) and (0, ±9)

l. (±4, 0) and (0, ±2)

m. Setting $y = 0$ yields the quadratic equation $x^2 = 100$, with solutions $x = \pm 10$. But setting $x = 0$ gives $-y^2 = 100$, or $y^2 = -100$, which has no solutions in $\mathbb{R}$. The intercepts are (±10, 0).

n. (0, ±11)

o. (0, 1)

Remember: In this class, every intercept is written as an ordered pair.

2. Let $x = 12$ in the ellipse equation and you should end up with y^2 equals a negative number, resulting in no solutions in $\mathbb{R}$.

3. a. (4, 0) (–4, 0) (0, 2) (0, –2)

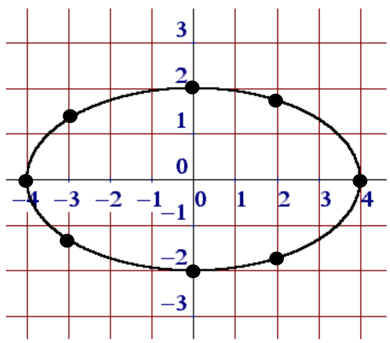
b. Letting $x = 2$ gives

$$\frac{2^2}{16}+\frac{y^2}{4} = 1 \;\Rightarrow\; \frac{1}{4}+\frac{y^2}{4} = 1 \;\Rightarrow\; \frac{y^2}{4} = \frac{3}{4}$$

$\Rightarrow\; y^2 = 3 \;\Rightarrow\; y = \pm\sqrt{3}$, which give us the two approximate points (2, 1.732) and (2, –1.732).

c. Letting $x = -3$ gives the points (–3, 1.323) and (–3, –1.323).

d.



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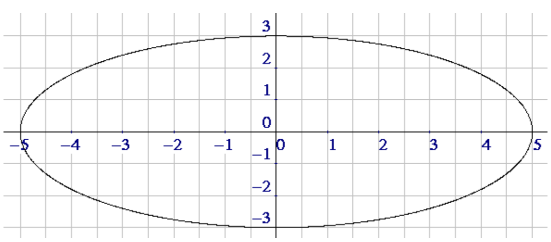

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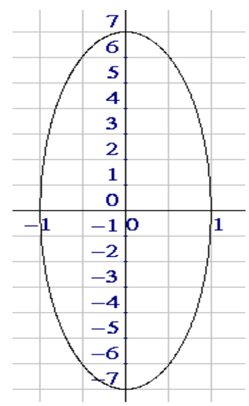

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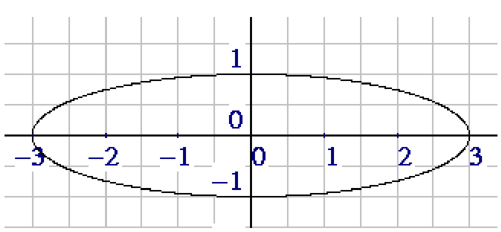

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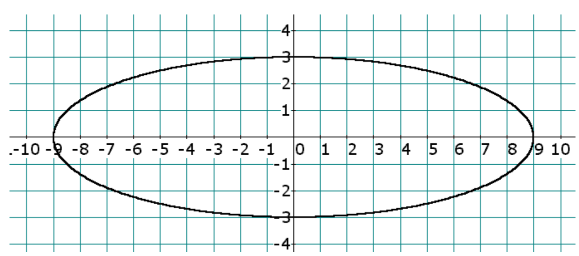

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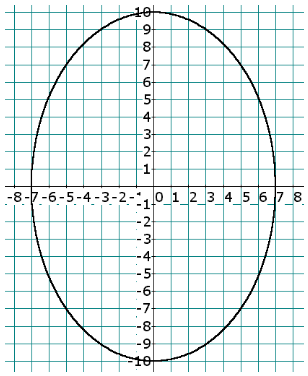

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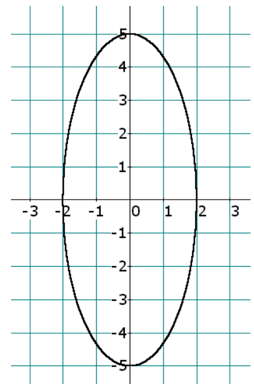


10. a. Neither variable is squared, so it's a line.

b. It's a vertical line.

c. It's a horizontal line.

d. One variable is squared; the other isn't. It's a parabola.

e. Both variables are squared, each with the same coefficient (1), so it's a circle.

f. Both variables are squared, but with different coefficients, but both positive; therefore, ellipse.

g. Both variables are squared, each with the same coefficient (1). It's a circle.

h. Both variables are squared, and the coefficients are 1/5 and 1/6; therefore, it's an ellipse.

i. Same conclusion has h.

j. Both variables are squared, and the coefficients are 17 and 22; therefore, it's an ellipse.

k. One variable is squared; the other isn't. It's a parabola.

l. Circle

m. Line

n. Both variables are squared, but their coefficients have opposite signs. So it neither a line, a parabola, a circle, nor an ellipse; it's a hyperbola.

11. a. Letting $x = 0$ in the ellipse equation results in $y = \pm 10$, which of course are real numbers. Therefore, 0 is allowed in the formula.

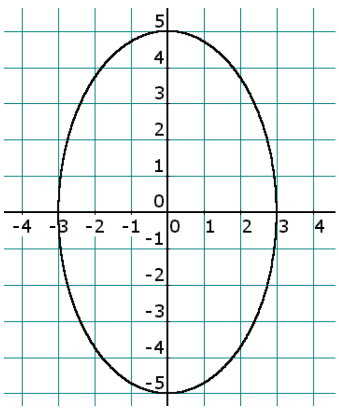
b. Letting $x = 4$ in the equation results in $y^2 = -\frac{700}{9}$, which has no solution in the real numbers. Thus, 4 is not allowed in the formula.

12. $(\pm 2, 0)$ (0 ± 5)

13. a. T b. F c. F

14. C, A, B, D, A

15.



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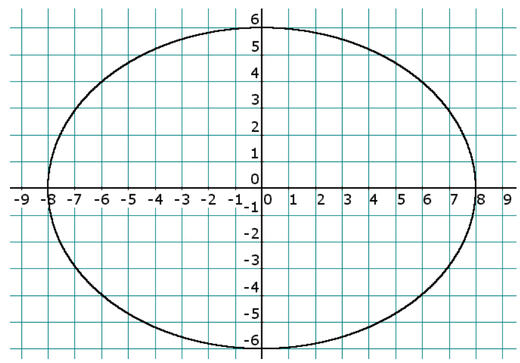

“Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air.”

– ***John Quincy Adams***